BULLETIN DES SCIENCES
ELSEVIER
www.elsevier.com/locate/bulsci

Stochastic control and compatible subsets of constraints ☆

Marc Quincampoix *, Catherine Rainer

Laboratoire de mathématiques, Unité CNRS UMR 6205, université de Bretagne occidentale, 6 Av. Le Gorgeu, 29200 Brest, France

Received 8 July 2004; accepted 9 July 2004

Available online 12 October 2004

Abstract

Given a stochastic differential control system and a closed set K in $\mathbb{R}^n$, we study the that, with probability one, the associated solution of the control system remains for ever in the set K. This set is called the *viability kernel of* K. If N is equal to the whole set K, K is said to be viable. We prove that, in the general case, the viability kernel itself is viable and we characterize it through some partial differential equations. We prove that, under suitable assumptions, also the boundary of N is viable. As an application, we give a new characterization of the value function of some optimal control problem.

Résumé

Etant donné un système de contrôle stochastique et un ensemble fermé K dans $\mathbb{R}^n$, nous étudions l'ensemble $N \subset K$ des points x, pour lesquels il existe un contrôle v tel que, avec probabilité 1, la trajectoire de la solution associée au système de contrôle reste pour toujours dans K. On appelle cet ensemble le noyau de viabilité de K. Lorsque $N = K$, on dit que K est viable. Nous montrons ici que, dans le cas général, le noyau de viabilité est viable et le caractérisons à l'aide d'une équation aux dérivées partielles. Nous montrons que, sous de bonnes hypothèses, le bord de N est gálement viable. Finalement, les résultats obtenus nous permettent de caractériser la fonction valeur d'un problème particulier de contrôle optimal.

☆ This work was supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00281 [Evolution Equations].

* Corresponding author.
E-mail address: msrc.quincampoix@univ-brest.fr (M. Quincampoix).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2004.07.001

MSC: 93E03; 64H20; 34F05; 49L25; 93E20

Keywords: Stochastic control with constraints; Stochastic viability; Optimal control

1. Introduction

Given a d-dimensional Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$, we consider a stochastic differential control system:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))\,dt + \sigma(X^{x,v(\cdot)}(t), v(t))\,dW(t), \quad t \in [0, \infty), \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Let K a closed set in $\mathbb{R}^n$. We say that the constraint K is compatible with the control system (1) – or following terminology of [2], we say that K is *viable* for (1) – if and only if for any $x \in K$, there exist a control v such that the corresponding solution to (1) satisfies

$$P\text{-almost surely}, \quad \forall t \geqslant 0, \quad X^{x,v(\cdot)}(t) \in K. \tag{2}$$

In general, the set K is not viable for (1). It is then natural to interest oneself on the set of initial conditions $x \in K$, for which it is possible to find a control $v(\cdot)$, such that (2) holds. We call this set the viability kernel of K and we denote it by $\mathrm{Viab}_{(b;\sigma)}(K)$. The main aim of the present paper is to study it.

When K is *viable* for the system (1), we have obviously $K = \mathrm{Viab}_{(b;\sigma)}(K)$. The property of viability of K for the system (1) has been extensively studied. We refer the reader to the monography [1] for the deterministic case. For the stochastic case, several characterizations have been obtained: through stochastic tangent cones in [2,3,13], through viscosity solutions of partial differential equations in [6,7]. We mention also [8] for time-depending constraints and [9] for viability for backward stochastic differential equations. This property has been also investigated in slightly different contexts in [4,5,16–20].

The viable kernel plays an important role in deterministic control, for example to study solutions of first order Hamilton Jacobi equation (see [10] for references).

The present paper is devoted to the stochastic case. Our first result gives an equivalent definition of the viability kernel of K: it is the largest closed subset of K which is viable.

We investigate fine properties of the boundary of the viability kernel, in particular that the boundary of the kernel itself is viable. This is a generalization of similar properties obtained for deterministic control in [21,22].

We also investigate an optimal stochastic control of the form

$$\inf_{v(\cdot)} \left(\text{ess-sup}_{\Omega}\, g\left(X^{x,v(\cdot)}(T)\right)\right). \tag{3}$$

We prove that the epigraph of the associated value function is the viability kernel of a suitably extended stochastic control system. As a consequence, we give some new characterizations of this value function and several other consequences.

The paper is organized as follows: After some preliminaries in the first section, we devote the second section to prove that the viability kernel is viable and to obtain useful consequences of this fact. Section 3 establishes properties of the boundary of the viability kernel. In Section 4, we apply previously obtained results to study the optimal control problem (3).

2. Preliminaries

Throughout the paper, by $\operatorname{Int} A$, $\operatorname{cl} A$ (or $\bar{A}$) and ∂A we denote respectively the interior, the closure and the boundary of a subset A of a metric space X. By $d_M(x)$ we denote a distance from a point x to a closed set M, by $\mathbb{1}_M$ we denote the indicator function of M. By $\operatorname{Argmin}_M \varphi$ (respectively $\operatorname{Argmax}_M \varphi$), we denote the set of local minima (respectively maxima) of φ with constraint M.

Let $(W(t), t \geqslant 0)$ be a d-dimensional standard Brownian motion on some complete probability space $(\Omega, \mathcal{F}, P)$ and $\nu = (\Omega, \mathcal{F}, P; W)$ the corresponding reference probability system. We denote by $(\mathcal{F}_t)_{t\geqslant 0}$ the natural filtration generated by W and augmented by the P-null sets of $\mathcal{F}$.

Let U be a compact metric space. We denote by $\mathcal{A} \equiv \mathcal{A}(\nu)$ the set of all U-valued processes $v(\cdot)$ which are progressively measurable with respect to $(\mathcal{F}_t)$. A process $v(\cdot) \in \mathcal{A}$ is called an admissible control.

We consider a stochastic control system described by the following stochastic differential equation:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))\,dt + \sigma(X^{x,v(\cdot)}(t), v(t))\,dW(t), & t \in [0, \infty), \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $b : \mathbb{R}^n \times U \to \mathbb{R}^n$; $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$.

The assumptions on b and σ are:

(H1) b and σ are uniformly continuous in (x, v);
(H2) $|\sigma(x, v) - \sigma(x', v)| \leqslant C_0|x - x'|,\ \forall x, x' \in \mathbb{R}^n, \forall v \in U$;
(H3) $\langle b(x, v) - b(x', v), x - x'\rangle \leqslant \mu|x - x'|^2,\ \forall x, x' \in \mathbb{R}^n, \forall v \in U$,

where $C_0 > 0$ and $\mu > 0$ are given constants.

Moreover we suppose that

(H4) the set $\{(\frac{1}{2}\sigma\sigma^*(x, v), b(x, v)), v \in U\}$ is convex and compact.

Let K be a given nonempty closed subset of $\mathbb{R}^n$. We define its viability kernel as follows:

Definition 2.1. The following subset N of K is called the *viability kernel of K*:

$$\operatorname{Viab}_{(b;\sigma)}(K) = \left\{x \in K, \exists \nu \text{ and } v \in \mathcal{A}(\nu), \text{ such that, } P\text{-a.s.}, \forall t \geqslant 0, X^{x,v(\cdot)}(t) \in K\right\}.$$

Let us recall that K is said to be viable for (4) if, for all $x \in K$, there exist a reference system ν and a process $v(\cdot) \in \mathcal{A}(\nu)$, such that, P-a.s., $X^{x,v}(t) \in K, \forall t \in [0, \infty)$. In this case, obviously, $\operatorname{Viab}_{(b;\sigma)}(K) = K$.

We recall a result that gives a necessary and sufficient criterium for viability through the following Hamilton–Jacobi–Bellman equation:

$$\inf_{v\in U,\sigma(x,v)^*\nabla u(x)=0} \mathcal{L}_{x,v}u = 0, \tag{5}$$

where we use the convention $\inf_\emptyset = +\infty$ and the notation

$$\mathcal{L}_{x,v}u = \langle \nabla u(x), b(x,v)\rangle + \frac{1}{2}\operatorname{tr}\bigl[D^2u(x)\sigma(x,v)\sigma^*(x,v)\bigr], \quad u \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}).$$

Theorem 2.1 [5, Theorem A1]. *The following assertions are equivalent*:

(i) *K enjoys viability with respect to* (4);
(ii) *The map $u(x) = 1 - \mathbb{1}_K(x)$ is a viscosity supersolution of* (5);
(iii) *For all function $\varphi \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ and $x \in \operatorname{Argmax}_K(\varphi)$,*

$$\inf_{v\in U,\sigma(x,v)^*\nabla\varphi(x)=0} \mathcal{L}_{x,v}\varphi \leqslant 0. \tag{6}$$

Remark 2.1. 1. In [6], it is proved that K is viable if and only if the distance to K $d_K(\cdot)$ is a viscosity supersolution to the following Hamilton Jacobi equation

$$\inf_{v\in U} \mathcal{L}_{x,v}u + d_K^2(x) - Cu(x) = 0,$$

where $C > 0$ is a constant large enough.

2. Assumptions (H1)–(H3) are classical to obtain the existence and uniqueness of solution to (4).

3. Assumption (H4) is used for obtaining a solution which is exactly viable (see [11]). Without (H4), we obtain a characterization of ε-viability, cf. Theorem 2 of [6].

We will also use a local version of this theorem, that is a slight transformation of a result of [7]: Consider a nonempty open set $\mathcal{O} \in \mathbb{R}^n$ and let the following Hamilton–Jacobi–Bellman equation:

$$\inf_{v\in U,\sigma(x,v)^*\nabla u(x)=0} \mathcal{L}_{x,v}u = 0, \quad x \in \mathcal{O}. \tag{7}$$

Theorem 2.2. *The following three assertions are equivalent*:

(i) *For all $x \in K \cap \mathcal{O}$, there exist ν and $v(\cdot) \in \mathcal{A}(\nu)$ such that, P-a.s., $X^{x,v}(t) \in K$, for all*

$$t \leqslant \tau^v(x) = \inf\bigl\{s \geqslant 0,\ X^{x,v}(s) \notin \mathcal{O}\bigr\};$$

(ii) *The map $u(x) = 1 - \mathbb{1}_K(x)$ is a viscosity supersolution of* (7);
(iii) *For all function $\varphi \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$ and $x \in \operatorname{Argmax}_{K\cap\mathcal{O}}(\varphi)$,*

$$\inf_{v\in U,\sigma(x,v)^*\nabla\varphi(x)=0} \mathcal{L}_{x,v}\varphi \leqslant 0.$$

Proof. (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are as in [5].

(ii) $\Rightarrow$ (i): Let $f:\mathbb{R}^n \to [0,1]$ be uniformly continuous and satisfy $(f(x)=0 \Leftrightarrow x \in K)$. Fix $C > 1$.

We prove exactly as in [7] that the application

$$V_o(x) = \inf_{\nu, v \in \mathcal{A}(\nu)} \int_0^{\tau^v(x)} \mathrm{e}^{-Cs} f\bigl(X^{x,v}(s)\bigr)\, ds$$

is the smallest nonnegative lower semi continuous (l.s.c. in short) supersolution of

$$\inf_{v \in U} \mathcal{L}_{x,v} V + f(x) - CV(x) = 0. \tag{8}$$

Now let $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \operatorname{Argmin}_{K \cap \mathcal{O}}(u - \varphi)$. Since u is a supersolution of (7), it holds that

$$\inf_{v \in U, \sigma(x,v)^* \nabla \varphi(x) = 0} \mathcal{L}_{x,v} \varphi \leqslant 0.$$

But, for $C > 1$, we have $f(x) - Cu(x) \leqslant 0$, for all $x \in \mathbb{R}^n$. Thus it holds also that

$$\inf_{v \in U} \mathcal{L}_{x,v} \varphi + f(x) - Cu(x) \leqslant 0.$$

This means that u is a supersolution of (8) and it follows that $u \geqslant V_o$. In particular, for $x \in K$, $V_o(x) = 0$. This implies (i). □

Set $G = \mathbb{R}^n \setminus \mathcal{O}$. In analogy to the deterministic case (see [23]), the property (i) of Theorem 2.2 can be called *viability with target* G. We will also use in the sequel the notion of viability kernel with target G:

Definition 2.2. The *viability kernel with target* G denoted by $\mathrm{Viab}_{(b;\sigma)}(K;G)$ is the following set:

$$\bigl\{x \in K, \exists \nu \text{ and } v \in \mathcal{A}(\nu),\ \text{such that, } P\text{-a.s.}, \forall 0 \leqslant t \leqslant \tau^v(x), X^{x,v}(t) \in K\bigr\},$$

where we set $\tau^v(x) = \inf\{s \geqslant 0, X^{x,v}(s) \in G\}$.

3. Properties of the viability kernel

A first important point to notice is that, if, we set

$$V(x) = \inf_{\nu, v \in \mathcal{A}(\nu)} E\left[\int_0^{+\infty} \mathrm{e}^{-Cs}(1 - \mathbb{1}_K)\bigl(X^{x,v}(s)\bigr)\, ds\right], \tag{9}$$

then, because the above infimum is a minimum (cf. [11]), the viability kernel can be written as

$$N := \mathrm{Viab}_{(b;\sigma)}(K) = \bigl\{x \in K, V(x) = 0\bigr\}. \tag{10}$$

Remark 3.1. In the formula (9), the function $1 - \mathbb{1}_K$ can be replaced by any nonnegative uniformly continuous function vanishing on K; for instance the distance function d_K. In this case, one can check that the function

$$x \mapsto \inf_{\nu, v \in \mathcal{A}(\nu)} E\left[\int_0^{+\infty} \mathrm{e}^{-Cs} d_K\big(X^{x,v}(s)\big)\,ds\right] \tag{11}$$

is Lipschitz. Therefore, by (10), $\mathrm{Viab}_{(b;\sigma)}(K)$ is closed.

We have to use in the sequel an equivalent characterization: the viability kernel N is the biggest subset of K such that, for any given reference system ν, for all $x \in N$, there exists a sequence of admissible controls $(v_\varepsilon)_{\varepsilon>0} \in \mathcal{A}(\nu)$ such that

$$E\left[\int_0^{+\infty} \mathrm{e}^{-Cs} d_K\big(X^{x,v_\varepsilon}(s)\big)\,ds\right] \leqslant \varepsilon.$$

The following proposition is a crucial point of the paper.

Proposition 3.1. *The viability kernel N is viable.*

Proof. Fix $x \in N$ hence $V(x) = 0$. Let (ν_0, v_0) an optimal control associated to x and (11). Hence, by an easy adaptation of Theoren 3.4 of [26], we obtain for any $\tau \geqslant 0$

$$V(x) = E\left[\int_0^{\tau} e^{-Cs} d_K\big(X^{x,v_0}(s)\big)\,ds + e^{-C\tau} V\big(X^{x,v_0}(\tau)\big)\right]$$

and consequently $E[V(X^{x,v_0}(\tau))] = 0$. Thus

$$X^{x,v_0}(\tau) \in N, \quad P\text{-a.s.}$$

Since this holds true for all $\tau > 0$, since $X^{x,v_0}(t)$ is continuous and N a closed set, it follows that we have also

$$X^{x,v_0}(\tau) \in N, \quad \forall \tau \geqslant 0, \quad P\text{-a.s.} \qquad \square$$

Corollary 3.1. *The viability kernel of K is the largest viable closed subset of K.*

We introduce now the following assumption:

(H5) For all $(p, A) \in \mathbb{R}^n \times \mathcal{S}$, $p \neq 0$, the map

$$x \in \mathbb{R}^n \mapsto \inf_{v \in U, \sigma(x,v)^* p = 0} \left(\big\langle b(x,v), p \big\rangle + \frac{1}{2} \operatorname{tr}\big(\sigma(x,v)\sigma^*(x,v)A\big) \right)$$

is continuous (with $\mathcal{S}$ the set of symmetric $n \times n$-matrices).

Remarks 3.1. 1. This assumption implies in particular that, for all $(p, A) \in \mathbb{R}^n \times \mathcal{S}$, $p \neq 0$, the set $\{v \in U, \sigma(x, v)^* p = 0\}$ is nonempty. Conversely, if the set is nonempty, the considered map is naturally l.s.c. So assumption (H5) means that it is also u.s.c.

2. Assumption (H5) seems rather restrictive. In particular, in the case without control, it is fulfilled if and only if $\sigma \equiv 0$, i.e. the dynamic is deterministic. As a nontrivial example where (H5) holds, we can cite the dynamic associated to the mean curvature motion in [5]. Remark also that a similar condition to (H5) is required in [24,25] to study stochastic target problems.

Proposition 3.2. *Suppose that* (H5) *holds. Set $\widehat{N} = cl(K \setminus N)$. We have, for all function $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in (\partial N \setminus \partial K)$ local maximum of φ in $\widehat{N}$,*

$$\sup_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi \leqslant 0.$$

Before proving Proposition 3.2, we need a technical result:

Lemma 3.1. *Let K be a closed set in $\mathbb{R}^n$, $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and x a strict local maximum of φ in K, such that $D^2\varphi(x)$ is invertible.*

Suppose that $\nabla\varphi(x) = 0$. Then at least one of the following assertions holds true:

(i) *There exists $w \in \mathbb{R}^n \setminus \{0\}$, a sequence $(\alpha_n)_{n\in\mathbb{N}} \searrow 0$ and $(x_n)_{n\in\mathbb{N}} \subset K$ converging to x, such that, for all $n \in \mathbb{N}$, $\varphi_n := \varphi + \alpha_n \langle w, \cdot - x\rangle$ has a local maximum in K at x_n and $\nabla\varphi_n(x_n) \neq 0$.*

(ii) *$D^2\varphi(x) \leqslant 0$.*

Proof. Suppose (i) false. Fix $z \in \mathbb{R}^n \setminus \{0\}$ and set $w = -D^2\varphi(x)z$. Consider $(\alpha_n)_{n\in\mathbb{N}} \searrow 0$ and x_n local maximum of φ_n. Then $\nabla\varphi_n(x_n) = 0$. Because x is a strict maximum for φ in K, one easily obtains $\lim_n x_n = x$. This implies the relation $\nabla\varphi(x_n) = \nabla\varphi_n(x_n) - \alpha_n w = -\alpha_n w$.

Thus, since $D^2\varphi(x)$ is invertible, by the theorem of local inversion, for sufficiently large $n \in \mathbb{N}$, we can write $x_n = (\nabla\varphi)^{-1}(-\alpha_n w) \in K$ and

$$(\nabla\varphi)^{-1}(-\alpha_n w) = (\nabla\varphi)^{-1}(0) + \left\langle \nabla\left((\nabla\varphi)^{-1}\right)(0), (-\alpha w)\right\rangle + \alpha_n \varepsilon(\alpha_n) \in K.$$

But $(\nabla\varphi)^{-1}(0) = x$ and $\nabla((\nabla\varphi)^{-1})(0) = (D^2\varphi(x))^{-1}$.

It follows that, for all $z \in \mathbb{R}^n \setminus \{0\}$, there exist $(\alpha_n)_{n\in\mathbb{N}} \searrow 0$ and $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ such that $z_n \to z$ and, for all $n \in \mathbb{N}$, $x_n := x + \alpha_n z_n \in K$.

Now let $z \in \mathbb{R}^n \setminus \{0\}$. Let $(\alpha_n)_{n\in\mathbb{N}} \searrow 0$ and $(z_n)_{n\in\mathbb{N}} \to z$, such that, for all $n \in \mathbb{N}$, $x + \alpha_n z_n \in K$. Since $x \in \mathrm{Argmax}_K\, \varphi$ and $\nabla\varphi(x) = 0$, we have, for all sufficiently large $n \in \mathbb{N}$,

$$\varphi(x) > \varphi(x + \alpha_n z_n) = \varphi(x) + \frac{1}{2}\alpha_n^2 \left\langle D^2\varphi(x) z_n, z_n\right\rangle + \alpha_n^2 \varepsilon(\alpha_n).$$

The result follows. □

Proof of Proposition 3.2. 1. We investigate first the case where φ and x are such that $\nabla\varphi(x) \neq 0$. For $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, set

$$h_\varphi(x) = \inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi.$$

We shall argue by contradiction: Suppose that the result of Proposition 3.2 is false. This means that there exists $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in (\partial N \setminus \partial K) \cap \mathrm{Argmax}_{\widehat{N}}(\varphi)$ such that $\nabla\varphi(x) \neq 0$ and $h_{-\varphi}(x) < -\alpha$, with $\alpha > 0$.

Suppose that φ has a strict local maximum in x (if it has not, we replace φ by $y \mapsto \varphi(y) - a|y - x|^2$, with $a > 0$ sufficiently small to be still in contradiction with the proposition).

Suppose that the assumption (H5) holds. Then, possibly after some modification of φ outside an open neighborhood of x, we can find some $\varepsilon > 0$ small enough, such that, for all $y \in \widehat{N}$,

$$\varphi(y) \geqslant \varphi(x) - \varepsilon \Rightarrow h_{-\varphi}(y) \leqslant -\alpha/2.$$

To get a contradiction with the definition of the viability kernel N, or more precisely, with the maximality of N, we will show that the set $N_\varepsilon = N \cup \{y \in \widehat{N}, \varphi(y) \geqslant \varphi(x) - \varepsilon, \nabla\varphi(y) \neq 0\}$ is viable. To do this, we will show that N_ε satisfies assertion (iii) of Theorem 2.1.

Let $\bar\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $\bar x \in \mathrm{Argmax}_{N_\varepsilon}(\bar\varphi)$.

- If $\bar x \in \mathrm{int}(N_\varepsilon)$, the necessary optimality conditions are $\nabla\bar\varphi(\bar x) = 0$ and $D^2\bar\varphi(\bar x) \leqslant 0$. Hence $h_{\bar\varphi}(\bar x) \leqslant 0$.
- If $\bar x \in \partial N \cap \partial N_\varepsilon$, we have $h_{\bar\varphi}(\bar x) \leqslant 0$, because N is viable.
- If $\bar x \in \partial N_\varepsilon \setminus \partial N$, we have

$$\bar x \in \mathrm{Argmax}_{N_\varepsilon}(\bar\varphi) \Rightarrow \bar x \in \mathrm{Argmax}_{\{y \in \widehat{N}, -\varphi(y) \leqslant -\varphi(x) + \varepsilon, \nabla\varphi(y) \neq 0\}}(\bar\varphi).$$

It follows by the classical second order necessary optimality conditions (cf. [15] for instance) that there exists $\lambda \geqslant 0$ such that

$$\begin{cases} \nabla\bar\varphi(\bar x) - \lambda\nabla(-\varphi)(\bar x) = 0, \\ \langle D^2\bar\varphi(\bar x)d, d\rangle - \lambda\langle D^2(-\varphi)(\bar x)d, d\rangle \leqslant 0, \quad \text{for } d \in \mathbb{R}^n, \\ \quad \text{such that } \langle \nabla(-\varphi)(\bar x), d\rangle = 0. \end{cases}$$

Thus

$$\begin{aligned} h_{\bar\varphi}(\bar x) &= \inf_{v \in U, \sigma(\bar x, v)^* \nabla\bar\varphi(\bar x) = 0} \left(\big\langle b(\bar x, v), \nabla\bar\varphi(\bar x) \big\rangle + \frac{1}{2} \mathrm{Tr}\big(D^2\bar\varphi(\bar x)\sigma(\bar x, v)\sigma^*(\bar x, v)\big) \right) \\ &\leqslant \lambda \inf_{v \in U, \sigma(\bar x, v)^* \nabla(-\varphi)(\bar x) = 0} \left(\big\langle b(\bar x, v), \nabla(-\varphi)(\bar x) \big\rangle \right. \\ &\quad \left. + \frac{1}{2} \mathrm{tr}\big(D^2(-\varphi)(\bar x)\sigma(\bar x, v)\sigma^*(\bar x, v)\big) \right) \\ &= \lambda h_{-\varphi}(\bar x) \leqslant -\lambda\alpha/2 \leqslant 0. \end{aligned}$$

This closes part 1 of the proof.

2. It remains to consider the case where $\nabla\varphi(x) = 0$. Modulo changing $\varphi(\cdot)$ in $\varphi(\cdot) - a|\cdot - x|^2$, for $a > 0$ sufficiently small, we can suppose again that x is a strict local maximum for φ in $\widehat{N}$ and that $D^2\varphi(x)$ is invertible. Now we can apply Lemma 3.1, and only the case where (i) occurs is not trivial.

Let $w \in \mathbb{R}^n \setminus \{0\}$, $\alpha_n \searrow 0$, $\varphi_n := \varphi + \alpha_n \langle w, \cdot - x\rangle$ and $x_n, n \in \mathbb{N}$ converging to x such that, for all $n \in \mathbb{N}$, $x_n \in \operatorname{Argmax}_{\widehat{N}} \varphi_n$, with $\nabla\varphi_n(x_n) \neq 0$. Since step 1, we can already apply Proposition 3.2 to all φ_n and x_n: for all $n \in \mathbb{N}$,

$$\sup_{v\in U, \sigma(x_n,v)^*\nabla\varphi_n(x_n)=0} \mathcal{L}_{x_n,v}\varphi_n \leqslant 0. \tag{12}$$

By assumptions (H2) and (H3), for all $\varepsilon > 0$, we can find $N \in \mathbb{N}$, such that, for all $n \geqslant N$ and all $v \in U$,

$$\mathcal{L}_{x,v}\varphi \leqslant \mathcal{L}_{x_n,v}\varphi + \varepsilon/2$$

and

$$\alpha_n\big\langle w, b(x_n, v)\big\rangle \geqslant -\varepsilon/2.$$

Further, by (12),

$$\mathcal{L}_{x_n,v}\varphi + \alpha_n\big\langle w, b(x_n, v)\big\rangle = \mathcal{L}_{x_n,v}\varphi_n \leqslant 0.$$

It follows that, for all $\varepsilon > 0$,

$$\sup_{v\in U, \sigma(x,v)^*\nabla\varphi(x)=0} \mathcal{L}_{x,v}\varphi \leqslant \varepsilon.$$

The result follows. □

As a consequence of Proposition 3.2, we can state a characterization of N through its indicator function.

Recall that the Epigraph of a real valued function φ is the set $\operatorname{Epi}(\varphi) = \{(x, y), \varphi(x) \leqslant y\}$. Similarly the Hypograph is $\operatorname{Hypo}(\varphi) = \{(x, y), \varphi(x) \geqslant y\}$. The upper semicontinuous envelope u^* is defined by: The hypograph of u^* is the closure of the hypograph of u: $\overline{\operatorname{Hypo} u} = \operatorname{Hypo} u^*$. The lower semicontinuous envelope u_* is defined by $\overline{\operatorname{Epi} u} = \operatorname{Epi} u_*$. Now u is a discontinuous viscosity solution of (13) if and only if u^* is a subsolution of (13) and u_* is a supersolution of (13).

Proposition 3.3. 1) *The viability kernel N of K is the largest closed subset H of K such that $1 - \mathbb{1}_H$ is a supersolution of the Eq.* (5).

2) *Suppose that* (H5) *holds. Then $u(x) = 1 - \mathbb{1}_N(x)$ is a discontinuous viscosity solution of*

$$\inf_{v\in U, \sigma(x,v)^*\nabla u(x)=0} \mathcal{L}_{x,v}u = 0, \quad x \in \mathbb{R}^n \setminus \partial K. \tag{13}$$

Proof. 1) By Proposition 3.1, the viable kernel N is viable. Thus Theorem 2.1 applies to N: in particular, u is a supersolution of (13). The assertion 1) follows directly from Corollary 3.1.

2) Remark first that we have to test only the points $x \in \partial N \setminus \partial K$. Let $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \partial N \setminus \partial K$, such that

$$\forall y \in \mathbb{R}^n, \quad u^*(y) - \varphi(y) \leqslant u^*(x) - \varphi(x).$$

Since on $\widehat{N}$, $u^* \equiv 1$, it holds in particular, for $y \in \widehat{N}$,

$$\varphi(y) \geqslant \varphi(x).$$

This means that $x \in \mathrm{Argmax}_{\widehat{N}}(-\varphi)$, and we can apply Proposition 3.2 to the map $-\varphi$:

$$\sup_{v \in U, \sigma(x,v)^* \nabla(-\varphi)(x)=0} \mathcal{L}_{x,v}(-\varphi) \leqslant 0,$$

or, equivalently,

$$\inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x)=0} \mathcal{L}_{x,v}\varphi \geqslant 0.$$

In view of Theorem 2.1(iii), this closes the proof. □

4. On the boundary of the viability kernel

In the deterministic case (see [22]), it is an already known result that the boundary of the viability kernel is viable with target ∂K. We shall see in this section that the result still holds in the stochastic case, provided that assumption (H5) is fulfilled.

Contrary to the deterministic case, it seems not possible to reason only on the trajectories, and the approach we present here is more analytic than probabilistic.

Theorem 4.1. *Suppose that assumption* (H5) *holds. Then, for all $x \in \partial N \setminus \partial K$, we can find ν and $v(\cdot) \in \mathcal{A}(\nu)$, such that P-a.s., the trajectories of $X^{x,v}$ stays on the boundary of N until they hit the boundary of K, i.e. there exist ν, and $v(\cdot) \in \mathcal{A}(\nu)$ such that*

$$X^{x,v}(t) \in \partial N, \quad \textit{for all } t \leqslant \tau^v(x),$$
$$\textit{with } \tau^v(x) = \inf\{s \geqslant 0, X^{x,v}(s) \notin \mathrm{int}(K)\}, \quad P\textit{-a.s.}$$

Before proving the theorem, we present two examples. The first one shows that, without assumption (H5), Theorem 4.1 is wrong. The second shows that, contrary to the deterministic case and even under assumption (H5), the set $\widehat{N}$ is not locally invariant[1] in $\mathrm{int}(K)$.

Example 1. Let the following stochastic system:

$$\begin{cases} dX^{x,v}(t) = dX^x(t) = (X^x(t) \wedge 0)\, dW(t) + dt, \\ X^x(0) = x \in \mathbb{R}, \end{cases} \tag{14}$$

where W is a real Brownian motion. Set $K = [-1, +\infty)$. Its viability kernel for (14) is $N = [0, +\infty)$: Indeed,

[1] A set M is said to be locally invariant in $\mathrm{int}(K)$, if, for all $x \in M \cap \mathrm{int}(K)$, all ν and $v \in \mathcal{A}(\nu)$, P-a.s., $X^{x,v}$ stays in M since it leaves $\mathrm{int}(K)$.

– for $x \in [0, +\infty)$, $t \geqslant 0$, $X^x(t) = x + t \in [0, +\infty)$,
– for $x \in K \setminus [0, +\infty)$, set $T_x = \inf\{s, X^x(s) \notin K\} \wedge \inf\{s, X^x(s) \geqslant 0\}$. Then it holds that $P[T_x < +\infty, X^x_{T_x} \notin [0, +\infty)] > 0$.

But ∂N is not viable: $\partial N = \{0\}$ and, $\forall t \geqslant 0$, $X^0(t) = t$.

Remark now that, for all $\varphi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, if $x \geqslant 0$, then

$$\inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi = \nabla\varphi(x),$$

and, if $x < 0$ with $\nabla\varphi(x) \neq 0$, then

$$\inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi = \inf \emptyset = +\infty.$$

Thus (H5) fails.

Example 2. In this example, $\widehat{N}$ is not locally invariant in $\mathrm{int}(K)$. Consider a stochastic differential control system as in (4), with $d = n$ and U a compact subset of $\mathbb{R}^m$ for some $m \geqslant 1$. Let $K \subset \mathbb{R}^n$ such that the viability kernel N of K for (4) is nonempty.

Now let $v_0 \in \mathbb{R}^m \setminus U$ and set $U' = U \cup \{v_0\}$. Set

$$\forall x \in \mathbb{R}^n, \quad \sigma(x, v_0) = I_n, \quad b(x, v_0) = 0.$$

Then σ and b satisfy (H1)–(H3) on $\mathbb{R}^n \times U'$.

Further we have, for all function $\varphi \in C^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$,

$$\nabla\varphi(x) \neq 0 \Rightarrow \sigma(x, v_0)^* \nabla\varphi(x) \neq 0$$
$$\Rightarrow \inf_{v \in U', \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi = \inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi.$$

It follows from Proposition 3.3, that N is still the viability kernel of K.

Now remark that, for all $x \in \mathbb{R}^n$, X^{x,v_0} is a $\mathbb{R}^n$-valued Brownian motion starting from x. Set $\tau^{v_0}(x) = \inf\{s \geqslant 0, X^{x,v_0}(s) \notin \mathrm{int}(K)\}$ and $T^{v_0}(x) = \inf\{s \geqslant 0, X^{x,v_0}(s) \in \mathrm{int}(N)\}$. It holds in particular that, if $x \in K \setminus N$, we have $P[T^{v_0}(x) > \tau^{v_0}(x)] > 0$.

Proof of Theorem 4.1. We will prove that ∂N satisfies condition (iii) of Theorem 2.2 with $\mathcal{O} = \mathbb{R}^n \setminus \partial K$ and have to test only points $\bar{x} \in \partial K \cap \mathrm{int}(K)$):

Let $\varphi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathrm{Argmax}_{\partial N \cap \mathrm{int}(K)} \varphi$. To simplify the writing, without lack of generality, we can set $\varphi(x) = 0$.

(I) Suppose first that $\nabla\varphi(x) \neq 0$. We have to consider three different possibilities.

1. If, around x, $N \subset \{\varphi \leqslant 0\}$, then $x \in \mathrm{Argmax}_N \varphi$. By Proposition 3.1, N is viable. Thus, by Theorem 2.1,

$$\inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi \leqslant 0.$$

2. If, around x, $\widehat{N} \subset \{\varphi \leqslant 0\}$, then $x \in \mathrm{Argmax}_{\widehat{N}} \varphi$. Thus, since Proposition 3.2,

$$\inf_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi \leqslant \sup_{v \in U, \sigma(x,v)^* \nabla\varphi(x) = 0} \mathcal{L}_{x,v}\varphi \leqslant 0.$$

3. It remains to show that there is no other case possible, i.e. we cannot have:

$$\forall V \in \mathcal{V}(x), \quad V \cap \{\varphi > 0\} \cap N \neq \emptyset \quad \text{and} \quad V \cap \{\varphi > 0\} \cap \widehat{N} \neq \emptyset, \tag{15}$$

where $\mathcal{V}(x)$ denotes the set of open neighborhoods of x.

Suppose by contradiction that (15) holds: For some sufficiently small $V \in \mathcal{V}(x)$, let $y \in V \cap \{\varphi > 0\} \cap N$ and $\hat{y} \in V \cap \{\varphi > 0\} \cap \widehat{N}$. Since $x \in \operatorname{Argmax}_{\partial N \cap \operatorname{int}(K)} \varphi$, we have,

$$\text{for } V \in \mathcal{V}(x) \text{ sufficiently small}, \quad V \cap \partial N \subset \{\varphi \leqslant 0\}. \tag{16}$$

This implies in particular that $y \in \operatorname{int}(N)$ (respectively $\hat{y} \in \operatorname{int}(\widehat{N})$).

Since $\nabla\varphi(x) \neq 0$, by the theorem of local inversion, we can find a diffeomorphism ψ from V to $\mathbb{R}^n$ such that

$$\begin{aligned}
&\psi\big(V \cap \{\varphi = 0\}\big) = \mathbb{R}^{n-1} \times \{0\},\\
&\psi\big(V \cap \{\varphi > 0\}\big) = \mathbb{R}^{n-1} \times (0, +\infty),\\
&\psi\big(V \cap \{\varphi < 0\}\big) = \mathbb{R}^{n-1} \times (-\infty, 0).
\end{aligned}$$

In particular, this implies that $V \cap \{\varphi > 0\}$ is connect by arcs: for all $t \in [0,1]$, $y_t := \psi^{-1}(t\psi(y) + (1-t)\psi(\hat{y})) \in V \cap \{\varphi > 0\}$.

Now set $\theta = \sup\{t \geqslant 0,\, y_t \notin \operatorname{int}(N)\}$. Since $y_1 = y \in \operatorname{int}(N)$ and $y_0 = \hat{y} \in \operatorname{int}(\widehat{N})$, it holds that $\theta \in (0,1)$. But $y_\theta \in N$ and $y_\theta \in \widehat{N}$. As a consequence, we get

$$V \cap \{\varphi > 0\} \cap \partial N \neq \emptyset. \tag{17}$$

This is in contradiction with (16).

(II) Suppose now that $\nabla\varphi(x) = 0$. Modulo some modifications on φ, we can suppose again that the assumptions of Lemma 3.1 are fulfilled. Consider $w \in \mathbb{R}^n, \alpha \searrow 0, \varphi$ and x_n as in Lemma 3.1. By part (I) of the proof, we have either $x_n \in \operatorname{Argmax}_N \varphi_n$ either $x_n \in \operatorname{Argmax}_{\widehat{N}} \varphi_n$ and in both cases

$$\inf_{v \in U, \sigma(x_n, v)^* \nabla\varphi(x_n) = 0} \mathcal{L}_{x_n, v}\varphi_n \leqslant 0.$$

We can conclude as in the proof of Proposition 3.2. □

5. Application to optimal control with supremal cost

We fix some time horizon $T > 0$ and, for all $(t,x) \in [0,T] \times \mathbb{R}^n$, we consider the dynamic

$$\begin{cases} dX^{t,x,v}(s) = b(X^{t,x,v}(s), v(s))\,ds + \sigma(X^{t,x,v}(s), v(s))\,dW(s), & s \in [t,T],\\ X^{t,x,v}(t) = x \in \mathbb{R}^n, \end{cases} \tag{18}$$

where $b : \mathbb{R}^n \times U \to \mathbb{R}^n$; $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$ satisfy the assumptions (H1)–(H4) and W and $v(\cdot)$ are defined as in the previous chapters.

In this section, we consider the following value function

$$W(t,x) = \inf_{v, v \in \mathcal{A}(v)} \big(\text{ess-sup}_\Omega\, g\big(X^{t,x,v(\cdot)}(T)\big)\big).$$

We shall characterize the Epigraph[2] of W. Throughout this section we shall suppose that $g : \mathbb{R}^n \mapsto \mathbb{R}$ is bounded and lower semicontinuous.

Theorem 5.1. *The Epigraph of W is the viability kernel of*

$$\mathcal{K} = [0, T] \times \mathbb{R}^n \times \mathbb{R}$$

with target $T \times \mathrm{Epi}(g)$, for the dynamic

$$\begin{cases} dS^{t,x,v}(s) = ds, \\ dX^{t,x,v}(s) = b(X^{t,x,v}(s), v(s))\,ds + \sigma(X^{t,x,v}(s), v(s))\,dW(s), \\ dY^{t,x,v}(s) = 0, \quad s \in [t, T], \\ S^{t,x,v}(t) = t \in [0, \infty), \quad X^{t,x,v}(t) = x \in \mathbb{R}^n, \quad Y^{t,x,v}(t) = y \in \mathbb{R}. \end{cases} \tag{19}$$

In short:

$$\mathrm{Epi}(W) = \mathrm{Viab}_{((1,b,0);(0,\sigma,0))}\big([0, T] \times \mathbb{R}^n \times \mathbb{R}; \{T\} \times \mathrm{Epi}(g)\big).$$

Proof. Let $(t, x, y) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}$. $(t, x, y) \in \mathrm{Viab}_{(1,b,0);(0,\sigma,0)}([0, T] \times \mathbb{R}^n \times \mathbb{R}; \{T\} \times \mathrm{Epi}(g))$ means that, P-a.s.,

$$\big(s, X^{t,x,v}(s), y\big) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}, \quad \text{on } \big\{s \leqslant \tau^v(x)\big\},$$

where

$$\begin{aligned} \tau^v(x) &= \inf\big\{r \geqslant t, \big(r, X^{t,x,v}(r), y\big) \in \{T\} \times \mathrm{Epi}(g)\big\} \\ &= \begin{cases} T & \text{if } g(X^{t,x,v}(T)) \leqslant y, \\ +\infty & \text{else.} \end{cases} \end{aligned}$$

Clearly the only way to realize this condition is to get some ν, $v(\cdot) \in \mathcal{A}(\nu)$, such that

$$g\big(X^{t,x,v}(T)\big) \leqslant y, \quad P\text{-a.s.}$$

This is equivalent to

$$\text{ess-sup}_\Omega\, g\big(X^{t,x,v}(T)\big) \leqslant y.$$

And this implies that

$$W(t, x) = \inf_{\nu, v \in \mathcal{A}(\nu)} \text{ess-sup}_\Omega\, g\big(X^{t,x,v}(T)\big) \leqslant y,$$

what means that (t, x, y) is in the Epigraph of W.

Conversely, let $(t, x, y) \in \mathrm{Epi}(W)$. By the very definition of W there exits a sequence $(X^{x,v_p}, p \in \mathbb{N})$ with

$$W(t, x) = \lim_p \big(\text{ess-sup}_\Omega\, g\big(X^{t,x,v_p(\cdot)}(T)\big)\big) \leqslant y.$$

[2] The Epigraph of a real valued function φ is the set $\mathrm{Epi}(\varphi) = \{(x, y), \varphi(x) \leqslant y\}$. Similarly the Hypograph is $\mathrm{Hypo}(\varphi) = \{(x, y), \varphi(x) \geqslant y\}$.

By classical arguments, there exists a subsequence $(X^{x,v_{p_k}}, k \in \mathbb{N})$, a probability space $(\Omega, \mathcal{F}, P)$, a Brownian motion W on this space, and an admissible control $v \in \mathcal{A}(\Omega, \mathcal{F}, P; W)$, such that $X^{x,v_{p_k}} \xrightarrow{\mathcal{D}} X^{x,v}$ uniformly on all compacts. Hence, we have, P-a.s,

$$g\big(X^{t,x,v}(T)\big) \leqslant y,$$

thus $(t, x, y) \in \mathrm{Viab}_{(1,b,0);(0,\sigma,0)}([0, T] \times \mathbb{R}^n \times \mathbb{R}; \{T\} \times \mathrm{Epi}(g))$. $\square$

We deduce several results from this theorem. First we get some characterizations of W.

Corollary 5.1. 1) *The application* $W : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is the smallest l.s.c. application* V *that satisfies*

(i) $V(T, x) = g(x)$, $x \in \mathbb{R}^n$;

(ii) $1 - \mathbb{1}_{\mathrm{Epi}(V)}$ *is a supersolution of*

$$\varphi_t + \inf_{v\in U, \sigma(x,v)^*\nabla_x\varphi(t,x,y)=0}\bigg(\big\langle \nabla_x\varphi(t, x, y), b(x, v)\big\rangle + \frac{1}{2}\operatorname{tr}\big[D^2_{xx}\varphi(t, x, y)\sigma(x, v)\sigma^*(x, v)\big]\bigg) = 0. \tag{20}$$

2) *Under assumption* (H5), $1 - \mathbb{1}_{\mathrm{Epi}(W)}$ *is a discontinuous viscosity solution of* (20).

3) *The application* $W : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *is the smallest l.s.c. supersolution of*

$$\varphi_t + \inf_{v\in U, \sigma(x,v)^*\nabla_x\varphi(t,x)=0} \mathcal{L}_{x,v}\varphi(t, \cdot) = 0. \tag{21}$$

4) *Under assumption* (H5), W *is a discontinuous viscosity solution of* (21).

5) *Suppose* (H5) *and assume that* g *is bounded and uniformly continuous. Then* W *is uniformly continuous with respect to* x *and the unique discontinuous viscosity solution of* (21) *with boundary condition* $W(T, x) = g(x)$, $x \in \mathbb{R}^n$.

Proof. 1), 2) First let us mention that $\mathrm{Epi}(W)$ is a viability kernel and thus a closed set. It follows that the application W is l.s.c. Now 1) and 2) are direct consequences of Proposition 5.1 and an easy generalization of Proposition 3.3 to viability kernels with target.

3) Fix (t, x) and φ such that $(t, x) \in \mathrm{Argmax}(\varphi - W)$ and $W(t, x) = \varphi(t, x)$. Define $\Psi : (s, \bar{x}, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \mapsto \varphi(s, \bar{x}) - y$. We claim that

$$\big(t, x, W(t, x)\big) \in \mathrm{Argmax}_{\mathrm{Epi}(W)} \Psi. \tag{22}$$

Indeed, since $(t, x) \in \mathrm{Argmax}(\varphi - W)$, we obtain, for all $(s, \bar{x}, y) \in \mathrm{Epi}(W)$,

$$\Psi\big(t, x, W(t, x)\big) = \varphi(t, x) - W(t, x) \geqslant \varphi(s, \bar{x}) - W(s, \bar{x}) \geqslant \varphi(s, \bar{x}) - y = \Psi(s, \bar{x}, y).$$

From Theorem 5.1, $\mathrm{Epi}(W)$ is viable, so applying condition (iii) of Theorem 2.2 to the system (19), we obtain

$$\varphi_t + \inf_{v\in U, \sigma(x,v)^*\nabla_x\varphi(t,x)=0} \mathcal{L}_{x,v}\varphi(t, \cdot) \leqslant 0.$$

Hence W is a supersolution to (21).

Further, consider $\overline{W}$ a l.s.c. supersolution to (21). Then one can check that $\mathrm{Epi}(\overline{W})$ is viable with target $\{T\} \times \mathrm{Epi}(g)$ for (19). So $\mathrm{Epi}(\overline{W})$ is included in the viability kernel which is equal to $\mathrm{Epi}(W)$. Hence $\overline{W} \geqslant W$.

4) Let us prove that W is a discontinuous viscosity solution. Since 3), it remains to prove that the upper semicontinuous envelope $W^\star$ is a subsolution of (21).

Fix (t,x) and φ such that $(t,x) \in \mathrm{Argmax}(-\varphi + W^\star)$ and $W^\star(t,x) = \varphi(t,x)$. Define $\Psi : (s,\bar{x},y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \mapsto -\varphi(s,\bar{x}) + y$. We prove as in 3) that

$$\big(t,x,W(t,x)\big) \in \mathrm{Argmax}_{\mathrm{Hypo}(W^\star)} \Psi.$$

Noticing that the Hamiltonian associated with the extended system (19) satisfies (H5), this yields by virtue of Proposition 3.2 (when N, (4) and ϕ are respectively replaced by Epi W, (19) and Ψ):

$$-\varphi_t + \sup_{v \in U, \sigma(x,v)^* \nabla_x \varphi(t,x) = 0} \mathcal{L}_{x,v}\big(-\varphi(t,\cdot)\big) \leqslant 0,$$

or equivalently

$$\varphi_t + \inf_{v \in U, \sigma(x,v)^* \nabla_x \varphi(t,x) = 0} \mathcal{L}_{x,v}\varphi(t,\cdot) \geqslant 0.$$

So $W^\star$ is a subsolution.

5) According to [14] Theorem 4.1, Eq. (21) has a unique discontinuity viscosity solution which is uniformly continuous in x. Thus the assertion follows from 4). □

Remark that using the viability of $\mathrm{Epi}(W)$ we obtain: For all $(t,x) \in [0,T] \times \mathbb{R}^n$, for all $s \in [t,T]$,

$$\inf_{\nu, v \in \mathcal{A}(\nu)} \text{ess-sup}_\Omega W\big(s, X^{t,x,v}(s)\big) \leqslant W(t,x), \quad P\text{-a.s.}$$

This is one side of the dynamic programming concerning W (see [24,25]).

The local viability of $\partial\,\mathrm{Epi}(W)$ leads also to a surprising corollary:

Corollary 5.2. *Suppose that W is continuous and that* (H5) *holds. Then, for all $(t,x) \in [0,T] \times \mathbb{R}^n$, there exist ν, $v(\cdot) \in \mathcal{A}(\nu)$, such that,*

$$\textit{for all } s \in [t,T], \quad W(s, X^{t,x,v}(s)) = W(t,x), \quad P\textit{-a.s.} \tag{23}$$

In particular,

$$P\textit{-a.s.}, \quad g\big(X^{t,x,v}(T)\big) = W(t,x). \tag{24}$$

Proof. Fix $(t,x) \in [0,T] \times \mathbb{R}^n$. Under assumption (H5), we can apply Theorem 4.1: since $(t,x,W(t,x)) \in \partial(\mathrm{Epi}(W))$, we can find ν, $v(\cdot) \in \mathcal{A}(\nu)$, such that, P-a.s.,

$$\text{for all } s \in [t,T], \quad (s, X^{t,x,v}(s), y) \in \partial\big(\mathrm{Epi}(W)\big). \tag{25}$$

But, since W is continuous, $\partial(\mathrm{Epi}(W))$ is nothing but the graph of W. The result follows. □

Remark 5.1. 1. Note that the continuity of W comes from considerations on partial differential equation (21). When g is uniformly continuous, W is uniformly continuous in x (as already mentioned in Corollary 5.1). When b and σ do not depend on x – as in the curvature motion case (cf. Remark 5 below) – W is continuous by Theorem 4.9 in [14].

2. If $\sigma \equiv 0$, relation (24) becomes trivial and (23) is well-known in deterministic control.

3. Let us consider the case where the dynamic does not depend from any control. In this case the relation (24) means that, for some ordinary Itô process and any bounded l.s.c. function g, we get

$$g\big(X^{t,x}(T)\big) = \text{ess-sup}_{\Omega}\, g\big(X^{t,x}(T)\big), \quad P\text{-a.s.},$$

thus the variable $g(X^{t,x}(T))$ is P-a.s. deterministic. This could be surprising at the first glanc. But if σ does not depend from any control, (H5) is equivalent to $\sigma \equiv 0$: Finally there is no contradiction, and this example only illustrates the fact that Corollary 5.2 becomes clearly wrong, if one removes the assumption (H5).

4. An equivalent result is used in [5], namely that the process $X^{x,v(\cdot)}$ involved in the representation formula the mean curvature motion attains at some fixed times the different level sets of the function g. In this case (21) is the well-known mean curvature equation (cf. [12]):

$$\begin{cases} -W_t - \Delta W + \frac{\langle D^2 W.DW, DW\rangle}{|DW|^2} = 0 & \text{in } [0,T] \times \mathbb{R}^n, \\ W(T,\cdot) = g(\cdot) & \text{in } \mathbb{R}^n. \end{cases} \tag{26}$$

In [5] it is proved that the solution of the above partial differential equation is the value function W associated with the control system

$$dX^{t,x,v(\cdot)}(s) = \sqrt{2}v(s)\,dW(s)$$

for $v \in U := \{v \in \mathcal{S} \mid v \geqslant 0, I - v^2 \geqslant 0 \text{ and } \mathrm{Tr}(I - v^2) = 1\}$.

References

[1] J.-P. Aubin, Viability Theory, Birkhauser, 1992.

[2] J.-P. Aubin, G. Da Prato, Stochastic viability and invariance, Annali Scuola Normale di Pisa 27 (1990) 595–694.

[3] J.-P. Aubin, G. Da Prato, The viability theorem for stochastic differential inclusions, Stochastic Anal. Appl. 16 (1) (1998) 1–15.

[4] M. Bardi, R. Jensen, A geometric characterization of viable sets for controlled degenerate diffusions, Set-Valued Anal. 10 (2–3) (2002) 129–141.

[5] R. Buckdahn, P. Cardaliaguet, M. Quincampoix, A representation formula for the mean curvature motion, SIAM J. Math. Anal. 33 (4) (2001) 827–846.

[6] R. Buckdahn, S. Peng, M. Quincampoix, C. Rainer, Existence of stochastic control under state constraints, C. R. Acad. Sci. Paris Sér. 1 327 (1998) 17–22.

[7] R. Buckdahn, M. Quincampoix, C. Rainer, A. Rascanu, Stochastic control with exit time and constraints, application to small time attainability of sets, Appl. Math. Optim. 49 (2) (2004) 99–112.

[8] R. Buckdahn, M. Quincampoix, C. Rainer, A. Rascanu, Viability in moving sets for stochastic differential equation, Adv. Differential Equations 7 (9) (2002) 1045–1072.

[9] R. Buckdahn, M. Quincampoix, A. Rascanu, Stochastic viability for backward stochastic differential equations and applications to partial differential equations, Probab. Theory Related Fields 116 (2000) 485–504.

[10] P. Cardaliaguet, M. Quincampoix, P. Saint-Pierre, Numerical schemes for discontinuous value functions of optimal control, Set-Valued Anal. 8 (2000) 111–126.
[11] N. El Karoui, D. Hu Nguyen, M. Jeanblanc-Piqué, Compactification methods in the control of degenerate diffusions: existence of an optimal control, Stochastics 20 (1987) 169–219.
[12] L.C. Evans, J. Spruck, Motion of level sets by mean curvature I, J. Differential Geom. 33 (1991) 635–681.
[13] S. Gautier, L. Thibault, Viability for constrained stochastic differential equations, Differential Integral Equations 6 (1993) 1394–1414.
[14] Y. Giga, S. Goto, H. Ishii, M. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana Univ. Math. J. 40 (2) (1991) 443–470.
[15] A.D. Ioffe, V.M. Tikhomirov, Theory of Extremal Problems, Studies in Mathematics and its Applications, vol. 6, North-Holland, Amsterdam, 1979.
[16] L. Mazliak, A note on weak viability for controlled diffusion, Stat. Probab. Lett. 49 (4) (2000) 331–336.
[17] L. Mazliak, C. Rainer, Exact and possible viability for controlled diffusions, Stat. Probab. Lett. 62 (2) (2003) 155–161.
[18] M. Michta, A note on viability under distribution constraints, Discuss. Math. Algebra Stoch. Methods 18 (2) (1998) 215–225.
[19] A. Milian, A note on stochastic invariance for Itô equations, Bull. Pol. Acad. Sci. Math. 41 (2) (1993) 139–150.
[20] J. Motyl, Viability of set-valued Itô equation, Bull. Pol. Acad. Sci. Math. 47 (1) (1999) 91–103.
[21] M. Quincampoix, Frontières de domaines d'invariance et de viabilité pour des inclusions différentielles avec contraintes, C. R. Acad. Sci. Paris Sér. I 311 (1990) 904–914.
[22] M. Quincampoix, Differential inclusions and target problems, SIAM J. Control Optim. 30 (2) (1992) 324–335.
[23] M. Quincampoix, V. Veliov, Viability with a target: theory and applications, in: B. Cheshankov, M. Tododrov (Eds.), Applications of Mathematical Engineering, Heron Press, Sofia, 1998, pp. 47–58.
[24] H. Soner, N. Touzi, Dynamic programming for stochastic target problems and geometric flows, J. Eur. Math. Soc. 4 (2002) 201–236.
[25] H. Soner, N. Touzi, Stochastic target problems, dynamic programming, and viscosity solutions, SIAM J. Control Optim. 41 (2) (2002) 404–424.
[26] J. Yong, X.Y. Zhou, Stochastic Controls, Springer-Verlag, 1999.